Ring Theory 14

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Q> Let (R, δ) be a Euclidean pair. Now, let $S \subseteq R$ be a non-trivial δ-closed subring and let $a \in S$ is non-zero non unit. Now given $b \in S \setminus \{0_R\}$ there exists a unique $k \in \mathbb{Z}_{\geq 0}$ and unique $r_0, r_1, \dots, r_k \in S$ such that :-

(i) $r_k \neq 0_R$

(ii) $\delta(r_0), \delta(r_1), \dots, \delta(r_k) < \delta(a)$ and

(iii) $b = r_0 + r_1 a + r_2 a^2 + \dots + r_k a^k$

Ans:- $b \in S \setminus \{0_R\} \Rightarrow b = qb + r \Rightarrow \delta(r) < \delta(b)$

$\Rightarrow q = 1_R,\ r = 0_R$

$b = a a^{-1} b + 0$

$$\inf(\delta(b) \mid b \in S) = 0$$

$\delta(b) < \delta(a)$ if b is a unit, $\delta(b) = \delta(1_R)$

In this case $k = 0$ will do and $r_0 = b \Rightarrow$ all three conditions satisfied

Let $\delta(b) = \inf(\delta(r) \mid r \in S, \delta(r) > m)$, such a b is chosen

If $\delta(b) < \delta(a)$ then $k = 0$ and $r_0 = a$ still holds.

If $\delta(b) > \delta(a)$ we will get,

$$b = qa + r \text{ with } \delta(r) < \delta(a)$$

Now S is $\underline{\delta\text{-closed} \Rightarrow q, r \in S} \Rightarrow \delta(b) \geq \delta(a) > \delta(r)$

$$\underbrace{\delta(q) < \delta(qa)}_{\delta\text{-positivity}} = \delta(b - r) \leq \max\{\delta(b), \delta(r)\} = \delta(b)$$

So we will get,

$$q = t_0 + t_1 a + \cdots + t_k a^k$$

$$\Rightarrow \quad b = qa + r = r + t_0 a + t_1 a^2 + \cdots + t_k a^{k+1}$$

$\Rightarrow$ all thee conditions suffices

Q) Let (R, δ) be a Euclidean pair and S be a non trivial subring of R. Then show that in S, Division Algorithm (unique quotient and remainder) holds true.

Ans:- $a \in S$, $\quad a = q_1 b + r_1 = q_2 b + r_2 \qquad \delta(r_1), \delta(r_2) < \delta(b)$

$$r_1 - r_2 = (q_1 - q_2) b$$

$$\delta((q_1 - q_2) b) \geq \delta(q_1 - q_2)$$

$$\delta(q_1 - q_2) \leq \delta(r_1 - r_2) \leq \max\{\delta(r_1), \delta(r_2)\} < \delta(b)$$

$$\Rightarrow \delta((q_1 - q_2) b) < \delta(b) \quad \Rightarrow \Leftarrow$$

$$\Rightarrow (q_1 - q_2) = 0 \quad \Rightarrow q_1 = q_2 \quad \Rightarrow r_1 = r_2$$

Q) If an ideal $I \subseteq R$ is free as an R-module, then I is a Principal ideal. A principal ideal I is free if it is generated by a non zero divisor. In general if R is an integral domain, then an ideal is free iff it is principal. (R is commutative for all cases)

Ans :- Let I be a free R-module,

If $I = 0$ we are done.

If $I \neq 0$, suppose it is not principal.

$\Rightarrow$ I have basis B and $x, y \in B$, $B = \{b_i\}$

The elements of B will not be zero divisors

Let $b_1, b_2 \in B$ and $b_1 \neq b_2$.

Rb_1 and Rb_2 will not be same

$b_1 b_2 - b_2 b_1 = 0$ $\Rightarrow \Leftarrow$ $\Rightarrow$ I is principal

If I is principal, $\Rightarrow$ $I = \langle a \rangle$ $\Rightarrow$ I is free

a is non-zero